

Three Simple Algorithms to Generate Kochen-Specker Sets with Thirty Six, Thirty Eight and Forty Rays in Three-Qubit System

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Abstract

We put forward three simple algorithms to generate Kochen-Specker sets used for parity proof of Kochen-Specker theorem in three-qubit system. These algorithms enable us to generate 320, 640 and 64 Kochen-Specker sets with 36, 38 and 40 rays, respectively.

Keywords: Kochen-Specker theorem; Contextuality; Hidden variable; Three-qubit.

1 Introduction

Besides nonlocality, contextuality is also a fascinating property of quantum mechanics. In the contextual quantum world, the outcome of a measurement is depends on which compatible observable might be measured together. A landmark statement of contextuality is the Kochen-Specker (KS) theorem which asserts that quantum mechanics can be completed only by contextual hidden variable model. Specifically, KS theorem states that in a Hilbert space of dimension $d > 2$, it is impossible to associate definite numerical values, 1 or 0, with every rays (vectors) in a finite set, in such a way that, (1) no two orthogonal rays are both assigned the value 1, and (2) no any complete basis are assigned the value 0 to all its rays.

The first explicit proof of KS theorem in 1967 used 117 rays [1]. This number of rays is greatly reduces to 33 and 24 for three and four dimensions, respectively [2]. Competition is on conceiving proof with the lowest number of rays. The

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current records for three-, four-, five-, six-, seven- and eight-dimensional systems are 31 rays [3], 18 rays [4], 29 rays [5], 31 rays [5], 34 rays [5] and 36 rays [6], respectively.

There are also some magnificent progresses on the experiment of testing KS theorem. In 1999, Meyer and Kent [7, 8] started a discussion on the possibility of testing KS theorem experimentally, for the proof proposed theoretically relied on infinite precision of measurement. By taking into account the unavoidable imprecisions exist in actual experiments, Cabello [9] proposed a KS inequality that satisfied by noncontextual hidden-variables theory but violated by quantum mechanics. Subsequently, the violation of the KS inequality has been shown in the experiments using trapped ions [10], single photons [11] and nuclear magnetic resonance system [12]. The inequality introduced in [9] is universal, i.e., any KS set in dimension $d \geq 3$ can be converted into a noncontextuality inequality which is state-independently violated [13].

Efforts of physicists no longer restricted on finding various proofs of KS theorem, the connection between the KS and Bell's theorems becomes one of the interesting research topics. Contextuality is a more general concept than nonlocality and it has been shown that every KS set can derive a maximally violated Bell inequality [14]. The state-independent noncontextuality inequalities provides a way to test the nonlocality of quantum system [15]. Besides, the investigation on KS theorem also extended to its application on quantum cryptography [16, 17].

Recently, there are some progresses on KS proofs in high dimension [18, 19]. For eight-dimensional three-qubit system, [18, 20] reported the numbers of parity proofs using 36, 38 and 40 rank-1 projectors, i.e., 320, 640 and 64, respectively. In this Letter, we will propose three simple algorithms to generate these KS sets. In some cases [4, 21, 22], the generation and investigation of huge number of KS sets rely on computer program, our algorithms has advantage of generating 1024 KS sets quick without involving any computer calculation and can be followed easily even by non-physicists. The similar beauty of simplicity is presented in [23] for four-dimensional KS sets.

This Letter is organized as follows. Algorithms of generating eight dimensional KS sets using 36, 38 and 40 rays are explicated in Section 2.1, Section 2.2 and Section 2.3, respectively. One example is given for each of the algorithm. The numbers of KS sets that can be generated by these algorithms are also explained in these sections. Conclusions are given in Section 3.

2 Three Algorithms

A Kochen-Specker (KS) set is a set of rays and bases used to prove KS theorem. A KS set can be used to perform parity proof if the number of bases is odd and each ray occurs an even number of times among the bases [20]. Table 3 in [20] gives types and numbers of KS sets in three-qubit system that can be used for

performing parity proof. Adopting the Algorithms I, II and III proposed in this section, the three different types of the KS sets can easily be generated.

Table 1 shows all the 25 bases used in the proof of Kernaghan and Peres [6] for a system of three qubits. The bases are formed by the 40 rays denoted as R_i , with $i = 1 - 40$, as shown in Table 2 [6]. Table 1 is the reproduction of Table 2 in [20]. The first 5 bases are pure bases ($PB_i, i = 1 - 5$) and the rest are hybrid bases ($HB_i, i = 6 - 25$). Hybrid bases formed by the equal mixture of rays from a pair of pure bases [20]. As easily seen from Table 1, the eight dimensional 40 rays occurs once in PB and 4 times in HB .

The three simple algorithms we propound in this section generate bases from Table 1 to form KS sets. To generate means to pick appropriate bases from Table 1 that satisfy prescribed conditions in the algorithms. Abstraction is necessary to explicate various cases of KS sets generation, thus it is difficult to understand by only reading various symbols in the algorithms. However this difficulty can be minimized by working out the examples provided, i.e., readers are encouraged to pick, copy the required bases and rays from Table 1 on a piece of paper and check the expected features accordingly.

The algorithms explicated in this Letters require 4 rays, $\Gamma^{ij} = \{\alpha, \beta, \gamma, \delta\}$, from PB that repeat its 3-element subsets $\Gamma_1^{ij} = \{\alpha, \beta, \gamma\}$, $\Gamma_2^{ij} = \{\alpha, \beta, \delta\}$, $\Gamma_3^{ij} = \{\alpha, \gamma, \delta\}$ and $\Gamma_4^{ij} = \{\beta, \gamma, \delta\}$ in HB . We name such property possessed by a specific set of 4 rays as Property I. There are 8 sets of 4-rays having Property I in each PB and all the 40 Γ^{ij} with $i = 1 - 5$ (subscript of PB) and $j = 1 - 8$ (subscript of the four-rays, fr) are listed in Table 3. Note that the elements of Γ^{ij} are listed in the ascending order. The complement of Γ^{ij} with respect to the PB_i is $\neg\Gamma^{ij} = \{\alpha', \beta', \gamma', \delta'\}$, again the rays are listed in ascending order. In our algorithms, steps proposed based on the sequence, k of the rays repeating 4 times, denoted as Σ_k , be chosen. All the ray in Γ^{ij} chosen are Σ_k , but the converse is not true.

Due to the Property I, each time of choosing Γ^{ij} will generate 1 PB and 4 HB s, we name such consequence of choosing Γ^{ij} as Property II. Both Properties I and II guarantee the 4 rays in Γ^{ij} occur 4 times, whereas the rays of $\neg\Gamma^{ij}$ in corresponding PB_i repeat its second time in the HB s generated. The rays that occur once in the 4 HB s are labeled as Λ_i , each of them has 4 elements and they play important role in determining Σ_k in the subsequent steps. For instance, picking $\Gamma^{34} = \{R17, R21, R22, R23\}$ from PB_3 generates bases shown in Table 4. The boldface rays in HB_{19} , HB_{20} and HB_{21} and HB_7 constitute Γ_1^{34} , Γ_2^{34} , Γ_3^{34} and Γ_4^{34} , respectively. The rays $\neg\Gamma^{34} = \{R18, R19, R20, R24\}$ are italicized, and they occur second time in HB_{19} , HB_{20} and HB_{21} and HB_7 , respectively. The sets of Λ_i also clearly shown in Table 4, for example $\Lambda_{19} = \{R11, R12, R15, R16\}$.

When more than one Γ^{ij} are chosen, Property II might generate the same HB s. In this case, we will pick only once the overlapping HB s from Table 1. The number of newly generated HB s determines the number of set for Λ_i .

Table 1: The 25 Bases formed by the 40 eight dimensional rays.

Index	Rays in Basis							
1	1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15	16
3	17	18	19	20	21	22	23	24
4	25	26	27	28	29	30	31	32
5	33	34	35	36	37	38	39	40
6	1	2	3	4	13	14	15	16
7	1	2	5	6	21	22	23	24
8	1	3	5	7	29	30	31	32
9	1	4	6	7	37	38	39	40
10	2	3	5	8	33	34	35	36
11	2	4	6	8	25	26	27	28
12	3	4	7	8	17	18	19	20
13	5	6	7	8	9	10	11	12
14	9	10	13	14	19	20	23	24
15	9	11	13	15	27	28	31	32
16	9	12	14	15	34	36	38	39
17	10	11	13	16	33	35	37	40
18	10	12	14	16	25	26	29	30
19	11	12	15	16	17	18	21	22
20	17	19	21	23	26	28	30	32
21	17	20	22	23	35	36	37	39
22	18	19	21	24	33	34	38	40
23	18	20	22	24	25	27	29	31
24	25	28	30	31	33	36	37	38
25	26	27	29	32	34	35	39	40

These cases will happen in Algorithm II and Algorithm III.

2.1 Algorithm I: 36 rays

The following steps generate KS sets of the type $28_2 8_4 - 11_8$. The symbol means 28 rays occur twice and 8 rays occur 4 times in 11 bases [20]. The subscript 8 means there are 8 rays in each basis. The 11 bases consists of 1 *PB* and 10 *HBs*. The existence of 1 *PB* means that only 1 Γ^{ij} is required. Algorithm I shows how to pick out Σ_k ($k = 1 - 8$) in such a way that the others 28 rays in the 11 bases occur twice.

Step 1: Pick the first Γ^{ij} from Table 3.

Choosing Γ^{ij} means to choose $\Sigma_1 = \alpha$, $\Sigma_2 = \beta$, $\Sigma_3 = \gamma$ and $\Sigma_4 = \delta$. As mentioned previously, Γ^{ij} incurs 1 *PB* and 4 *HBs* (Property II). In addition, there are 4 rays of $-\Gamma^{ij}$ and 4 sets of Λ_i^n , with $n = 1$ here. The added superscript

Table 2: The 40 rays used in the proof of Kernaghan and Peres for a three-qubit system. The $\bar{1}$ means -1 .

1	10000000	9	11110000	17	11001100	25	10101010	33	1001011 0
2	01000000	10	11 $\bar{1}$ 10000	18	1100 $\bar{1}$ 100	26	1010 $\bar{1}$ 0 $\bar{1}$ 0	34	100 $\bar{1}$ 0110
3	00100000	11	1 $\bar{1}$ 110000	19	1 $\bar{1}$ 001 $\bar{1}$ 00	27	10 $\bar{1}$ 010 $\bar{1}$ 0	35	10010 $\bar{1}$ 10
4	00010000	12	1 $\bar{1}$ 110000	20	1 $\bar{1}$ 00 $\bar{1}$ 100	28	10 $\bar{1}$ 0 $\bar{1}$ 010	36	100 $\bar{1}$ 0 $\bar{1}$ 10
5	00001000	13	00001111	21	00110011	29	01010101	37	0110 $\bar{1}$ 001
6	00000100	14	000011 $\bar{1}$ $\bar{1}$	22	001100 $\bar{1}$ $\bar{1}$	30	01010 $\bar{1}$ 0 $\bar{1}$	38	01 $\bar{1}$ 01001
7	00000010	15	00001 $\bar{1}$ 1 $\bar{1}$	23	001 $\bar{1}$ 001 $\bar{1}$	31	010 $\bar{1}$ 010 $\bar{1}$	39	0 $\bar{1}$ 101001
8	00000001	16	00001 $\bar{1}$ $\bar{1}$ 1	24	001 $\bar{1}$ 00 $\bar{1}$ 1	32	010 $\bar{1}$ 0 $\bar{1}$ 01	40	0 $\bar{1}$ $\bar{1}$ 0 $\bar{1}$ 001

Table 3: Sets of four rays possessing Property I. These sets are all taken from pure bases.

fr	PB_1	PB_2	PB_3	PB_4	PB_5
1	1 2 3 5	9 10 11 13	17 18 19 21	25 26 27 29	33 34 35 40
2	1 2 4 6	9 10 12 14	17 18 20 22	25 26 28 30	33 34 36 38
3	1 3 4 7	9 11 12 15	17 19 20 23	25 27 28 31	33 35 36 37
4	1 5 6 7	9 13 14 15	17 21 22 23	25 29 30 31	33 37 38 40
5	2 3 4 8	10 11 12 16	18 19 20 24	26 27 28 32	34 35 36 39
6	2 5 6 8	10 13 14 16	18 21 22 24	26 29 30 32	34 38 39 40
7	3 5 7 8	11 13 15 16	19 21 23 24	27 29 31 32	35 37 39 40
8	4 6 7 8	12 14 15 16	20 22 23 24	28 30 31 32	36 37 38 39

n indicates that the corresponding Λ_i are produced after the execution of Step n .

As an example, by choosing Γ^{11} , we have $\Sigma_1 = \alpha = R1$, $\Sigma_2 = \beta = R2$, $\Sigma_3 = \gamma = R3$ and $\Sigma_4 = \delta = R5$. Due to Property II, we need to choose HB_6 , HB_7 , HB_8 and HB_{10} as well as PB_1 . Thus we obtain 5 bases after execution of Step 1. The set of rays that repeat twice in these 5 bases is $\neg\Gamma^{11} = \{R4, R6, R7, R8\}$. Bear in mind that rays in Λ_i^n occur once in the 5 bases picked based on Property II. Hints on choosing Σ_5 come from $\Lambda_6^1 = \{R13, \dots, R16\}$, $\Lambda_7^1 = \{R21, \dots, R24\}$, $\Lambda_8^1 = \{R29, \dots, R32\}$ and $\Lambda_{10}^1 = \{R33, \dots, R36\}$.

Step 2: Pick a ray in one of the Λ_i^1 as Σ_5 . Denote the corresponding Λ_i^1 as Λ_a^1 .

Since the ray Σ_5 must all occur in HB and one was already existed in the HB_a , there are only 3 new HB s generated after carried out the Step 2. Three pairs of rays, one pair from each of $\Lambda_i^1 (i \neq a)$, must repeat in the 3 newly generated HB s. We denote them as Δ_i , with the i taking the same subscript

Table 4: Example for explaining the important concepts of Γ^{ij} , $\neg\Gamma^{ij}$ and Λ_i in Algorithms I, II and III.

Index	Rays in Basis							
3	17	<i>18</i>	<i>19</i>	<i>20</i>	21	22	23	<i>24</i>
7	1	2	5	6	21	22	23	<i>24</i>
19	11	12	15	16	17	<i>18</i>	21	22
20	17	<i>19</i>	21	23	26	28	30	32
21	17	<i>20</i>	22	23	35	36	37	39

as in Λ^1 .

In our example, we choose $R13$ from Λ_6^1 to generate HB_{14} , HB_{15} and HB_{17} . Note that $\Delta_7 \subset \Lambda_7^1 = \{R23, R24\}$, $\Delta_8 \subset \Lambda_8^1 = \{R31, R32\}$ and $\Delta_{10} \subset \Lambda_{10}^1 = \{R33, R35\}$ occur also in HB_{14} , HB_{15} and HB_{17} , respectively.

Step 3: Pick one ray from Δ_i as Σ_6 .

As Σ_6 is chosen from Δ_i , it was existed in pair. Thus Step 3 generates only 2 HB s. In our example, we execute Step 3 by taking the $R23$ from Δ_7 as Σ_6 . The second $R23$ existed in HB_{14} . The third and fourth $R23$ exist in HB_{20} and HB_{21} , thus HB_{20} and HB_{21} are generated.

Step 4: Consider any one of the remaining two Δ_i , only one ray of it generates a new HB . Take it as Σ_7 .

We choose to consider Δ_8 in our example. Thus, $R32$ is Σ_7 for it generates only HB_{25} , the first 3 rays of $R32$ existed in HB_8 , HB_{15} and HB_{20} .

Step 5: Pick those ray from the last Δ_i that generates no new HB as Σ_8 .

It can be easily checked that the Σ_8 in our example that generates no any HB is $R35$ from Δ_{10} .

In our example, the 11 bases generated are PB_1 , HB_6 , HB_7 , HB_8 , HB_{10} , HB_{14} , HB_{15} , HB_{17} , HB_{20} , HB_{21} and HB_{25} (follow order in which they are generated). The 8 rays occur 4 times are $R1$, $R2$, $R3$, $R5$, $R13$, $R23$, $R32$ and $R35$ (follow order in which they are chosen). The remaining 28 rays occur twice in the generated 11 bases.

For Algorithm I, there are 40 ways of choosing the first 4 Σ_i from Γ^{ij} in Step 1, 4 ways of choosing Σ_5 in Step 2 and 2 ways of choosing Σ_6 in Step 3, therefore the total number of KS sets in the form of $28_2 8_4 - 11_8$, N_I that can be generated is $40 \times 4 \times 2 = 320$. Note that because Σ_7 and Σ_8 in Step 4 and Step 5 are restricted by the Σ_6 chosen in Step 3, they don't contribute to N_I .

2.2 Algorithm II: 38 rays

This section explicates steps to generate KS sets of the type $24_2 14_4 - 13_8$. The 13 bases consists of 3 PBs and 10 HBs . Algorithm II shows how to pick out

the 14 rays that occurs 4 times, Σ_k with $k = 1 - 14$, in such a way that the remaining 24 rays occur twice.

Step 1: Pick any one of the Γ^{ij} from Table 3.

This step in fact is the same as Step 1 in Algorithm I. Due to Step 1, 1 PB_i contains rays $\{\alpha, \beta, \gamma, \delta\}$ and 4 HBs contains rays $\{\alpha, \beta, \gamma\}$, $\{\alpha, \beta, \delta\}$, $\{\alpha, \gamma, \delta\}$ and $\{\beta, \gamma, \delta\}$ are chosen. The 4 rays in Γ^{ij} are labeled as $\Sigma_1 - \Sigma_4$. The rays in $\neg\Gamma^{ij}$ repeat twice and the second time occur in the HBs generated. Again, we have 4 sets of rays, Λ_i^1 .

We take the same example as in Step 1 of Algorithm I.

Step 2: Pick the second Γ^{ij} based on one of the Λ_i^1 .

By adding one appropriate ray, it is easily seen from Table 3 that any 3 rays from Λ_i^1 offer us options to choose the second Γ^{ij} . Denotes the chosen Λ_i^1 as Λ_a^1 . Label the 4 rays in the second Γ^{ij} as $\Sigma_5 - \Sigma_8$. Three Λ_i^2 obtained as Step 2 generates only 3 HBs . Define non-empty sets $\Xi_j^i = \Lambda_i^1(i \neq a) \cap \Lambda_j^2$ and its complement as $\neg\Xi_j^i$. Note that Ξ_j^i and $\neg\Xi_j^i$ have 2 and 4 elements, respectively.

In our example, by adding adding $R9, R10, R11$ or $R12$, $\Lambda_6^1 = \{R_{13}, \dots, R_{16}\}$ allows us to choose $\Sigma_5 - \Sigma_8$ either from $\Gamma^{24}, \Gamma^{26}, \Gamma^{27}$, or Γ^{28} , respectively. We now take Γ^{24} to generate 1 PB and 3 HBs , i.e., PB_2, HB_{14}, HB_{15} and HB_{16} . Note that the overlapping HB is HB_6 itself. We get, by inspection, $\Xi_{14}^i = \{R_{23}, R_{24}\}$, $\Xi_{15}^i = \{R_{31}, R_{32}\}$ and $\Xi_{16}^{10} = \{R_{34}, R_{36}\}$. On the other hand, $\neg\Xi_{14}^i = \{R_{19}, \dots, R_{22}\}$, $\neg\Xi_{15}^i = \{R_{27}, \dots, R_{30}\}$ and $\neg\Xi_{16}^{10} = \{R_{33}, R_{35}, R_{38}, R_{39}\}$.

Step 3: Referring to Table 3, it is obvious that the rays from each pair of Ξ_j^i and $\neg\Xi_j^i$ offer two options of taking the third Γ^{ij} . Pick one of it.

Due to Step 3, 1 PB and 2 HBs are generated and 2 Λ_i^3 are obtained. Note that the number of sets for both Λ_i^1 and Λ_i^2 reduced from 3 to 2 after the execution of Step 3 and we denote them as $\Lambda_i^{1'}$ and $\Lambda_i^{2'}$, respectively.

In our example, we take Ξ_{14}^i and the rays $R19$ and $R21$ from $\neg\Xi_{14}^i$ to form Γ^{37} , thus $\Sigma_9 = R19$, $\Sigma_{10} = R21$, $\Sigma_{11} = R23$ and $\Sigma_{12} = R24$. Due to this choice, only PB_3, HB_{20} and HB_{22} are generated, as HB_7 and HB_{14} that contain $\{R_{21}, R_{23}, R_{24}\}$ and $\{R_{19}, R_{23}, R_{24}\}$ were existed. Therefore, we have $\Lambda_8^{1'} = \{R_{29}, \dots, R_{32}\}$, $\Lambda_{10}^{1'} = \{R_{33}, \dots, R_{36}\}$, $\Lambda_{15}^{2'} = \{R_{27}, R_{28}, R_{31}, R_{32}\}$, $\Lambda_{16}^{2'} = \{R_{34}, R_{36}, R_{38}, R_{39}\}$, $\Lambda_{20}^3 = \{R_{26}, R_{28}, R_{30}, R_{32}\}$ and $\Lambda_{22}^3 = \{R_{33}, R_{34}, R_{38}, R_{40}\}$.

Step 4: Among the rays in $\Lambda_i^{1'}$, $\Lambda_i^{2'}$ and Λ_i^3 , two of them repeat three times and six of them occur once. These eight rays constitute the last base of the KS set generating in this algorithm.

By adding the last base, we obtain Σ_{13} and Σ_{14} . In our example, the rays that repeat 3 times in $\Lambda_8^{1'}$, $\Lambda_{10}^{1'}$, $\Lambda_{15}^{2'}$, $\Lambda_{16}^{2'}$, Λ_{20}^3 and Λ_{22}^3 are $R32$ and $R34$, while the rays that occur once are $R26, R27, R29, R35, R39$ and $R40$. Therefore, the last base generated from Step 4 is HB_{25} .

Thus, the 13 bases generated in our example are $PB_1, HB_6, HB_7, HB_8, HB_{10}, HB_2, HB_{14}, HB_{15}, HB_{16}, HB_3, HB_{20}, HB_{22}$ and HB_{25} (follow order

in which they are generated). The 14 rays that occur 4 times are $R1, R2, R3, R5, R9, R13, R14, R15, R19, R21, R23, R24, R32$ and $R34$ (follow order in which they are chosen). The remaining 24 rays occur twice in the generated 13 bases.

It can be seen from Algorithm II that to generate KS sets of the form $24_2 14_4 - 13_8$ require 3 Γ^{ij} s that determine $\Sigma_k (k = 1 - 12)$. It is clear from Step 3 that the number of choices for third Γ^{ij} is determined by the second Γ^{ij} chosen, where $i = 2, i = 3, i = 4$ (i is the index of bases, specifically for pure bases when used for Γ^{ij}) give 6, 4 and 2 options, respectively. On the other hand, there are 6 combinations of the first 2 Γ^{ij} , which can be listed in terms of i as 12, 13, 14, 23, 24 and 34. There are 8 ways of getting the first Γ^{ij} in Step 1 and 4 ways of getting second Γ^{ij} in Step 2, therefore the total number of KS sets in the type of $24_2 14_4 - 13_8$, N_{II} that can be generated by Algorithm II is $8 \times 4 \times [6 + 4 + 2 + 4 + 2 + 2] = 640$.

2.3 Algorithm III: 40 rays

This section explicates steps to generate KS sets of the type $20_2 20_4 - 15_8$. The 15 bases are composed of 5 PBs and 10 HBs . Algorithm III shows how to pick out the 20 rays that occurs 4 times, Σ_k with $k = 1 - 20$, in such a way that there are 20 rays occur twice in the KS sets generated.

Steps 1, 2 and 3: Same as Step 1, Step 2 and Step 3, respectively, of Algorithm II.

As an example, we take Γ^{11}, Γ^{24} and Γ^{37} for the $\Sigma_k (k = 1 - 12)$ as happened in the first 3 steps of example in Section 2.2. As a result, we obtain $\Lambda_8^{1'} = \{R29, \dots, R32\}$ and $\Lambda_{10}^{1'} = \{R33, \dots, R36\}$, $\Lambda_{15}^{2'} = \{R27, R28, R31, R32\}$ and $\Lambda_{16}^{2'} = \{R34, R36, R38, R39\}$, $\Lambda_{20}^3 = \{R26, R28, R30, R32\}$ and $\Lambda_{22}^3 = \{R33, R34, R38, R40\}$, respectively.

Step 4: Extract the fourth Γ^{ij} from $(\Lambda_i^{1'})_{4th}$, $(\Lambda_i^{2'})_{4th}$ and Λ_i^3 .

The sets of rays from $\Lambda_i^{1'}$, $\Lambda_i^{2'}$ and Λ_i^3 are subset to either the fourth or fifth unchosen PHs , and we indicate them by adding a subscript $4th$ or $5th$ accordingly. The fourth Γ^{ij} is given as $[(\Lambda_i^{1'})_{4th} \cap (\Lambda_i^{2'})_{4th}] \cup [(\Lambda_i^{1'})_{4th} \cap (\Lambda_i^3)_{4th}] \cup [(\Lambda_i^{2'})_{4th} \cap (\Lambda_i^3)_{4th}]$.

Execution of Step 4 generates the fourth PB and another HB .

In our example, $\Gamma^{48} = [\Lambda_8^{1'} \cap \Lambda_{15}^{2'}] \cup [\Lambda_8^{1'} \cap \Lambda_{20}^3] \cup [\Lambda_{15}^{2'} \cap \Lambda_{20}^3]$. Thus, only PB_4 and HB_{24} are newly generated bases, whereas HB_8, HB_{15} and HB_{20} had been generated in the previous steps.

Step 5: Extract the fifth Γ^{ij} from $(\Lambda_i^{1'})_{5th}$, $(\Lambda_i^{2'})_{5th}$ and Λ_i^3 .

The fifth Γ^{ij} is given as $[(\Lambda_i^{1'})_{5th} \cap (\Lambda_i^{2'})_{5th}] \cup [(\Lambda_i^{1'})_{5th} \cap (\Lambda_i^3)_{5th}] \cup [(\Lambda_i^{2'})_{5th} \cap (\Lambda_i^3)_{5th}]$.

Execution of Step 5 generates the fifth PB and none of HB .

In our example, $\Gamma^{52} = [\Lambda_{10}^{1'} \cap \Lambda_{16}^{2'}] \cup [\Lambda_{10}^{1'} \cap \Lambda_{22}^3] \cup [\Lambda_{16}^{2'} \cap \Lambda_{22}^3]$. Thus, PB_5 is the only generated base, as $HB_{10}, HB_{16}, HB_{22}$ and HB_{24} had been generated

in the previous steps.

Thus, the 15 bases generated in our example are $PB_1, HB_6, HB_7, HB_8, HB_{10}, HB_2, HB_{14}, HB_{15}, HB_{16}, HB_3, HB_{20}, HB_{22}, HB_4, HB_{24}$ and HB_5 (follow order in which they are generated). The 20 rays that occur 4 times are $R_1, R_2, R_3, R_5, R_9, R_{13}, R_{14}, R_{15}, R_{19}, R_{21}, R_{23}, R_{24}, R_{28}, R_{30}, R_{31}, R_{32}, R_{33}, R_{34}, R_{36}, R_{38}$ (follow order in which they are chosen). The remaining 20 rays occur twice in the generated 15 bases.

Algorithm III requires 5 Γ^{ij} s to generate KS sets of the form $20_2 20_4 - 15_8$. The 5 Γ^{ij} s determine $\Sigma_k (k = 1 - 20)$. The numbers of ways of choosing first, second and third Γ^{ij} in Step 1, Step 2 and Step 3 are 8, 4 and 2, respectively. There is only one way of choosing fourth and fifth Γ^{ij} once the third Γ^{ij} is chosen in Step 3. Thus the total number of KS sets in the type of $20_2 20_4 - 15_8$, N_{III} that can be generated using Algorithm III is $8 \times 4 \times 2 = 64$.

3 Conclusion

The numbers of parity proof of Kochen-Specker (KS) theorem in eight dimensional system using 36, 38 and 40 rays are reported recently in [18, 20]. We put forward three simple algorithms to generate all of them in the type of $28_2 8_4 - 11_8$, $24_2 14_4 - 13_8$ and $20_2 20_4 - 15_8$, respectively. Each of the algorithm is explained with the aid of an example, and the numbers of KS sets that can be generated using these algorithms are 320, 640 and 64, respectively, which are square with the numbers reported in [18, 20].

As our algorithms do not involve any computer calculation, they are very helpful for those who lack of programming skill to generate all the 1024 KS sets or for those who need only few KS sets anytime and anywhere. The sets of 4-rays listed in Table 3 play a crucial role in our algorithms and some steps of picking the 4-rays are overlap in the three algorithms. It is clear that the simplicity of the algorithms to generate four-dimensional KS sets shown in [23] do exist in the case of eight-dimensional KS sets as well.

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